

A Quantitative Version of the Gibbard-Satterthwaite Theorem for Three Alternatives

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Abstract

The Gibbard-Satterthwaite theorem states that every non-dictatorial election rule among at least three alternatives can be strategically manipulated. We prove a quantitative version of the Gibbard-Satterthwaite theorem: a random manipulation by a single random voter will succeed with a non-negligible probability for any election rule among three alternatives that is far from being a dictatorship and from having only two alternatives in its range.

1 Introduction

A Social Choice Function (SCF), or an election rule, aggregates the preferences of all members of a society towards a common social choice. The study of SCFs dates back to the works of Condorcet in the 18th century, and has expanded greatly in the last decades.

One of the obviously desired properties of an SCF is *strategy-proofness*: a voter should not gain from voting strategically, that is, from reporting false preferences instead of his true preferences (such voting is called in the sequel *manipulation*). However, it turns out that this property cannot be obtained by any reasonable SCF. This was shown in a landmark theorem of Gibbard [Gib73] and Satterthwaite [Sat75]:

Theorem 1.1 (Gibbard, Satterthwaite). *Any SCF which is not a dictatorship (i.e., the choice is not made according to the preferences of a single voter), and has at least three alternatives in its range, can be strategically manipulated.*

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The Gibbard-Satterthwaite theorem implies that we cannot hope for full truthfulness in the context of voting, since any reasonable election rule can be manipulated. However, it still may be that such a manipulation is possible only very rarely, and thus can be neglected in practice.

In this paper we prove a *quantitative* version of the Gibbard-Satterthwaite theorem in the case of three alternatives, showing that if the SCF is not very close to a dictatorship or to having only two alternatives in its range, then even a random manipulation by a randomly chosen voter will succeed with a non-negligible probability. Thus, one cannot hope that manipulations will be negligible for any reasonable election rule.¹

In order to present our results we need a few standard definitions. First we formally define an SCF and a profitable manipulation.

Definition 1.2. An SCF on n voters and m alternatives is a function $F : (L_m)^n \rightarrow \{1, 2, \dots, m\}$, where L_m is the set of linear orders on m alternatives. A set of preferences given by the voters, i.e., $(x_1, x_2, \dots, x_n) \in (L_m)^n$, is called a *profile*. When we want to single out the i th voter, we write the profile as (x_i, x_{-i}) , where x_{-i} denotes the preferences of the other voters.

A *profitable manipulation* by voter i at the profile (x_1, \dots, x_n) is a preference $x'_i \in L_m$, such that $F(x'_i, x_{-i})$ is preferred by voter i (according to his “true” preference order x_i) over $F(x_i, x_{-i})$. A profile is called *manipulable* if there exists a profitable manipulation for some voter at that profile.

Now we define the quantitative settings we consider. Throughout the paper, we make the *impartial culture assumption* [Bla58], meaning that the profiles are distributed uniformly.² Under the uniform probability measure, the distance of F from a dictatorship is simply the fraction of values that has to be changed in order to turn F into a dictatorship. Similarly, in the case of three alternatives, the distance of F from having only two alternatives in its range is the minimal probability that an alternative is elected.

We quantify the probability of manipulation in the following way:

Definition 1.3. The *manipulation power* of voter i on an SCF F , denoted by $M_i(F)$, is the probability that x'_i is a profitable manipulation of F by voter i at profile (x_1, \dots, x_n) , where x_1, \dots, x_n and x'_i are chosen uniformly at random from L_m .

¹We note that functions that are very close to being a dictatorship may have a very small number of manipulable profiles (see e.g. [MPS04]). However, all of the prominent SCFs are far from being a dictatorship.

²Note that we cannot hope for an impossibility result for *every* distribution, e.g. since for every SCF one may consider a distribution on its non-manipulable profiles.

We note that our notion of *manipulation power* resembles notions of power and influence which play important roles in voting theory and in theoretical computer science. Specifically, our reliance on the uniform probability distribution makes our notion analogous to the *Banzhaf Power Index* from voting theory [Fis73], and to Ben-Or and Linial's notion of *influence* [BL85]. While the latter two notions coincide for monotone Boolean functions, the notion of Ben-Or and Linial deals also with the influence of coalitions (i.e., larger sets of voters) and with much more general protocols for aggregation. Similarly, the notion of manipulation power of one and more voters can be of interest in much greater generality.

Finally, we define a notion related to Generalized Social Welfare Functions which plays a central role in our proof.

Definition 1.4. A Generalized Social Welfare Function (GSWF) on n voters and m alternatives is a function $G : (L_m)^n \rightarrow \{0, 1\}^{\binom{m}{2}}$. That is, given the preference orders of the voters, G outputs the preferences of the society amongst each pair of alternatives.

A GSWF G satisfies the *Independence of Irrelevant Alternatives* (IIA) condition if the preference of the society amongst any pair of alternatives (A, B) depends only on the individual preferences between A and B , and not on other alternatives.

Now we are ready to state our main theorem.

Theorem 1.5. *There exist universal constants $C, C' > 0$ such that for every $\epsilon > 0$ and any n the following holds:*

- *If F is an SCF on n voters and three alternatives, such that the distance of F from a dictatorship and from having only two alternatives in its range is at least ϵ , then*

$$\sum_{i=1}^n M_i(F) \geq C \cdot \epsilon^6.$$

- *If, in addition, F is neutral (that is, invariant under permutation of the alternatives), then:*

$$\sum_{i=1}^n M_i(F) \geq C' \cdot \epsilon^2.$$

We note that the value of the constant C obtained in our proof of Theorem 1.5 is extremely low (see Remark 5.2), and thus the first assertion applies only in the asymptotic setting. Unlike the value of C , the obtained value of C' is reasonable.

The proof of the theorem consists of three steps:

1. **Reduction from low manipulation power to low dependence on irrelevant alternatives:** We show that if $\sum_{i=1}^n M_i(F)$ is small, then in some sense, the question whether the output of F is alternative A or alternative B depends only a little on alternatives other than A and B . Specifically, the probability of changing the outcome of F from A to B by altering the individual preferences between *all other alternatives* (and leaving the preferences between A and B unchanged) is low. This reduction is obtained by a directed isoperimetric inequality, which we prove using the FKG correlation inequality [FKG71] (or, more precisely, using Harris' inequality [Har60]).
2. **Reduction from an SCF with low dependence on irrelevant alternatives to a GSWF with a low paradox probability:** We show that given an SCF F on three alternatives with low dependence on irrelevant alternatives, one can construct a GSWF G on three alternatives which satisfies the IIA condition and has a low probability of paradox (a paradox occurs if for some profile, the society prefers A over B , B over C and C over A). Furthermore, the distance of G from dictatorship and from always ranking one alternative at the top/bottom is roughly the same as the distance of F from dictatorship and from having only two alternatives in its range, respectively.
3. **Applying a quantitative version of Arrow's impossibility theorem:** We use the quantitative versions of Arrow's theorem³ obtained by Kalai [Kal02] (in the neutral case), Mossel [Mos09], and Keller [Kel10] to show that since G has low paradox probability, it has to be close either to a dictatorship or to always ranking one alternative at the top/bottom. Translation of the result to F yields the assertion of the theorem. We note that the proofs of the quantitative Arrow theorem are quite complex and use discrete Fourier analysis on the Boolean hypercube and hypercontractive inequalities.

For a fixed value of ϵ , Theorem 1.5 implies lower bounds of $\Omega(1)$ and $\Omega(1/n)$ on $\sum_i M_i(F)$ and $\max_i M_i(F)$, respectively. It is easy to see that the lower bound on $\sum_i M_i(F)$ cannot be improved (up to the value of C and the dependence on ϵ), and that the lower bound on $\max_i M_i(F)$ cannot be improved to $\Omega(1)$. The latter follows since for the plurality SCF on n voters, only an $O(1/\sqrt{n})$ fraction of the profiles can be manipulated at all by any single voter, and thus $M_i(\text{Plurality}) = O(1/\sqrt{n})$ for all i . However, it is still possible that one can obtain a better lower bound than $\Omega(1/n)$ on $\max_i M_i(F)$, and we leave this as our first open problem.

Our second open problem concerns the case of more than three alternatives, $m > 3$. While some parts of our proof extend to this case (see Section 6), we were

³Arrow's theorem and its quantitative versions are described in Section 4.

not able to extend all required parts of the proof. After the preliminary version of this paper was written, several papers tried to resolve this case, and the most notable result is by Isaksson, Kindler, and Mossel [IKM09], who obtained a quantitative Gibbard-Satterthwaite theorem for $m > 3$ alternatives, under the only additional assumption of neutrality (see Theorem 2.1 below). However, the case of general SCFs on more than three alternatives is still open, and we leave it as our second open problem. We do conjecture that the theorem generalizes to $m > 3$, perhaps with the exact form of the bound decreasing polynomially in m (like the bound obtained by Isaksson et al. in the neutral case).

Our result can be viewed as part of the study of computational complexity as a barrier against manipulation in elections. A brief overview of the work in this direction, several follow-up results, and a short discussion of their implications is presented in Section 2. In Sections 3, 4, and 5 we present the three steps of our proof. Finally, we discuss the case of more than three alternatives in Section 6.

2 Related Work

Since the Gibbard-Satterthwaite theorem was presented, numerous works studied ways to overcome the strategic voting obstacle. The two best-known ways are allowing payments (see, e.g., [Gro73]) and restricting the voters' preferences (see [Mou80]).

Another way, suggested in 1989 by Bartholdi, Tovey, and Trick [BTT89], is to use a *computational barrier*. That is, to show that there exist reasonable SCFs for which, while a manipulation does exist, it cannot be found efficiently, and thus in practice, the SCF can be considered strategy-proof. Bartholdi et al. [BTT89] constructed a concrete SCF for which they proved that finding a profitable manipulation is *NP*-hard as an algorithmic problem. This approach was further explored by Bartholdi and Orlin [BO91] who proved that manipulation is *NP*-hard also for the well-known Single Transferable Vote (STV) election rule. In a related line of research, several papers showed that for various SCFs, the problem of *coalitional manipulation* (i.e., when a coalition of voters tries to coordinate their ballots in order to get their favorite alternative elected), is *NP*-hard for some SCFs, even for a constant number of alternatives (see [CS03, CSL07, EL05, FHH09, HHR07]).

However, while the results in this direction are encouraging, the computational barrier they suggest against manipulation may be practically insufficient. This is because all the results study the *worst case complexity* of manipulation, and show that manipulation is computationally hard for specific instances. In order to practically prevent manipulation, one should show that it is computationally hard for most instances, or at least in the *average case*.

In the last few years, several papers considered the hardness of manipulation in the average case [CS06, PR07, PR07b, XC08b, ZPR09].⁴ Their results suggest that unlike worst-case complexity, it appears that various SCFs can be manipulated relatively easily in the average case – that is, for an instance chosen at random according to some typical distribution. However, all these results consider specific families of SCFs, and manipulation by coalitions rather than by individual voters.

Our results also study manipulation in the “average case” by examining the success probability of a random manipulation by a randomly chosen voter, and yield a general impossibility result in the case of three alternatives, showing that for any reasonable SCF, such manipulation succeeds with a non-negligible probability. However, our result does not have direct implications on the study of computational hardness of manipulation, since in the case of a constant number of alternatives, the number of possible manipulations by a single voter is constant, and thus manipulation by a single voter cannot be computationally hard in this setting.

Follow-Up Work

Since the preliminary version of this paper [FKN08] appeared in FOCS’08, three follow-up works generalized its results to more than three alternatives, under various additional constraints.

The first follow-up work is by Xia and Conitzer [XC08], who use similar techniques to show that a random manipulation will succeed with probability of $\Omega(1/n)$ for any SCF on a constant number of alternatives satisfying the following five conditions:

1. Homogeneity – The outcome of the election does not change if each vote is replaced by k copies of it.
2. Anonymity – The SCF treats all the voters equally.
3. Non-Imposition – Any alternative can be elected.
4. Cancelling out – The outcome is not changed by adding the set of all possible linear orders of the alternatives as additional votes.
5. A complex stability condition (see [XC08] for the exact formulation).

While the conditions look a bit restrictive, Xia and Conitzer show that they hold for several well-known SCFs, including all positional scoring rules, STV, Copeland, Maximin, and Ranked Pairs.

⁴It should be noted that the success probability of a random manipulation for SCFs with a small number of voters and alternatives was studied a long time ago in a paper of Kelly [Kel93].

The second follow-up work is by Dobzinski and Procaccia [DP09]. They consider the case of two voters and an arbitrary number m of alternatives, and show that if an SCF is ϵ -far from a dictatorship and satisfies Pareto optimality (i.e., if both voters prefer alternative A over B , then B is not elected), then a random manipulation will succeed with probability at least ϵ/m^8 . The techniques used in the proof of [DP09] are relatively simple, and the authors suggest that possibly their result can be generalized to any number of voters, by modifying an inductive argument of Svensson [Sve99] that extends the proof of the classical Gibbard-Satterthwaite theorem from two voters to n voters, for any n .

The most recent, and most notable, work is by Isaksson, Kindler and Mossel [IKM09] who prove a quantitative version of the Gibbard-Satterthwaite theorem for a general number of alternatives, under the only additional assumption of neutrality.

Theorem 2.1 (Isaksson, Kindler, and Mossel). *Let F be a neutral SCF on $m \geq 4$ alternatives which is at least ϵ -far from a dictatorship. Consider a random manipulation generated by choosing a profile and a manipulating voter at random, and replacing four adjacent alternatives in the preference order of that voter by a random permutation of them. Then*

$$\Pr[\text{The manipulation is successful}] \geq \frac{\epsilon^2}{10^9 n^4 m^{34}}.$$

The techniques used by Isaksson et al. are combinatorial and geometric, and contain a generalization of the canonical path method which allows to prove isoperimetric inequalities for the interface of three bodies.

The result of Isaksson et al. shows that for any neutral SCF which is far from a dictatorship, a random manipulation by a single randomly chosen voter will succeed with a non-negligible probability. Thus, a single voter with black-box access to the SCF can find a manipulation efficiently.

However, this result still does not imply that the agenda of using computational hardness as a barrier against manipulation is completely hopeless, for three reasons:

1. The result relies on the assumption that the votes are distributed uniformly. It is possible to argue that in real-life situations, the distribution of the votes is far from uniform, and thus the result does not apply.
2. The result applies only to neutral SCFs.
3. While the result implies that with a non-negligible probability, a random manipulation by a randomly chosen voter succeeds, it is still possible that for

most of the voters, manipulation cannot be found efficiently (or even at all), while only for a polynomially small portion of the voters a manipulation can be found efficiently. Thus, it is possible that only a few voters can manipulate efficiently, while most voters cannot.

For an extensive overview of the study of computational complexity as a barrier against manipulation, and a further discussion on the implication of our results and the results of the follow-up works, we refer the reader to the survey [FP10] by Faliszewski and Procaccia.

3 Reduction from Low Manipulation Power to Low Dependence on Irrelevant Alternatives

In this section we show that if F is an SCF on three alternatives such that the manipulation power⁵ of the voters on F is small, then in some sense, the dependence of F on irrelevant alternatives is low. We quantify this notion as follows:

Definition 3.1. Let F be an SCF on three alternatives and let a, b be two alternatives. For a profile $x \in (L_3)^n$, denote by $x^{a,b} \in \{0, 1\}^n$ the vector which represents the preferences of the voters between a and b , where $x_i^{a,b} = 1$ if the i th voter prefers a over b , and $x_i^{a,b} = 0$ otherwise. The dependence of the choice between a and b on the (irrelevant) alternative c is:

$$M^{a,b}(F) = \Pr[F(x) = a, F(x') = b],$$

where $x, x' \in (L_3)^n$ are chosen at random, subject to the restriction $x^{a,b} = (x')^{a,b}$.

By the definition, $M^{a,b}(F)$ measures how often a change of the individual preferences between the alternative c and the alternatives a, b leads to changing the output of F from a to b or vice versa. Thus, the notion measures how much the (irrelevant) alternative c affects the question of whether a or b is elected.

We note that $M^{a,b}$ can be also viewed as measuring kind of a manipulation, where all the voters together attempt to change the output of F to be b rather than a by re-choosing at random all their preferences – except for those between a and b . However, this definition does not require that anyone in particular gains from changing the output.

The result we prove is the following:

⁵See Definition 1.3.

Lemma 3.2. Let F be an SCF on three alternatives. Then for every pair of alternatives a, b ,

$$M^{a,b}(F) \leq 6 \sum_{i=1}^n M_i(F).$$

In order to prove Lemma 3.2, we define a certain combinatorial structure and relate it both to $M^{a,b}(F)$ and to $\sum_i M_i(F)$.

We begin with a convenient way to represent a profile $x \in (L_3)^n$, given the individual preferences between a and b (denoted by $x^{a,b}$). Note that for any specific value $z^{a,b}$ of $x^{a,b}$, there are exactly 3^n possible values of x that agree with it. Indeed, the agreement of x with $z^{a,b}$ fixes the preferences of all voters between a and b in x , and each voter may choose one of three locations for c : above both a and b , below both of them, or between them. Thus, for every fixed $z^{a,b}$ we can view the set $\{x | x^{a,b} = z^{a,b}\}$ as isomorphic to $\{0, 1, 2\}^n = \{\text{above}, \text{between}, \text{below}\}^n$. We use $v = (v_1, \dots, v_n)$ to denote an element in this set. Thus, once $x_i^{a,b}$ is fixed, $v_i \in \{0, 1, 2\}$ encodes both $x_i^{b,c}$ and $x_i^{c,a}$. For example, $x_i^{a,b} = 0$ and $v_i = 0$ encodes the preference $c \succ_i b \succ_i a$.

Next, we define two sets which are closely related to the definition of $M^{a,b}(F)$.

Definition 3.3. For every value $z^{a,b}$ of the preferences between a and b , let

$$A(z^{a,b}) = \{x | x^{a,b} = z^{a,b}, F(x) = a\}, \text{ and}$$

$$B(z^{a,b}) = \{x | x^{a,b} = z^{a,b}, F(x) = b\}.$$

Both $A(z^{a,b})$ and $B(z^{a,b})$ are viewed as residing in the space $\{0, 1, 2\}^n$.

In terms of these definitions, we clearly have:

$$M^{a,b}(f) = \mathbb{E}_{x \in (L_3)^n} \left[\frac{|A(x^{a,b})|}{3^n} \cdot \frac{|B(x^{a,b})|}{3^n} \right]. \quad (1)$$

In order to relate $M_i(F)$ to the sets $A(x^{a,b})$ and $B(x^{a,b})$, we endow the set $\{0, 1, 2\}^n$ with a structure of a directed graph, whose edges correspond to (some of) the profitable manipulations by voter i . For each fixed value of $x^{a,b}$, for each $i \in \{1, 2, \dots, n\}$, and for each $v_{-i} \in \{0, 1, 2\}^{n-1}$, the graph has three directed edges in direction i between the possible values of v_i : $0 \rightarrow 1$, $1 \rightarrow 2$, and $0 \rightarrow 2$. The following definition counts the directed edges going “upward” from a subset of $\{0, 1, 2\}^n$.

Definition 3.4. Let $A \subseteq \{0, 1, 2\}^n$. The upper edge border of A in the i th direction, denoted by $\partial_i A$, is the set of directed edges in the i th direction defined above whose tail is in A and whose head is not in A . That is,

$$\partial_i A = \{(v_{-i}, v_i, v'_i) \mid (v_{-i}, v_i) \in A, (v_{-i}, v'_i) \notin A, v_i < v'_i\}.$$

The upper edge border of A is $\partial A = \bigcup_i \partial_i(A)$.

We now relate $M_i(F)$ to the upper edge borders in the i th direction of $A(x^{a,b})$ and $B(x^{a,b})$.

Lemma 3.5. For any $1 \leq i \leq n$, we have:

$$M_i(f) \geq \frac{1}{6} \cdot 3^{-n} \cdot \mathbb{E}_{x \in (L_3)^n} \left[|\partial_i A(x^{a,b})| + |\partial_i B(x^{a,b})| \right]. \quad (2)$$

Proof. We compute a lower bound on $M_i(F)$ by choosing x and x' at random, differing only (possibly) in the preferences of the i th voter, and providing a lower bound on the probability that the i th coordinate of x' is a profitable manipulation of x . We perform the random choice as follows: First we choose at random $x_{-i}^{a,b} \in \{0, 1\}^{n-1}$, $x_i^{a,b} \in \{0, 1\}$, and $x_i'^{a,b} \in \{0, 1\}$. With probability $1/2$, we have $x_i'^{a,b} = x_i^{a,b}$, and the rest of the analysis is conditioned on this event indeed occurring (a conditioning that does not affect the distribution chosen). We next choose $v_{-i} \in \{0, 1, 2\}^{n-1}$, and finally we choose $v_i \in \{0, 1, 2\}$ and $v_i' \in \{0, 1, 2\}$.

We claim that if $(v_{-i}, v_i, v_i') \in \partial_i A$, then either x_i' is a manipulation of x or x_i is a manipulation of x' .

Indeed, note that by the definition of $\partial_i A$, the condition $(v_{-i}, v_i, v_i') \in \partial_i A$ implies that when moving from x_i to x_i' , voter i lowered his relative preference of c without changing his ranking of the pair (a, b) , and this changed the output of F from a to some other result $t \in \{b, c\}$. We have two possible cases:

1. If, according to x_i , voter i prefers t to a , then x_i' is a manipulation of x .
2. If x_i ranks a above t , then this is definitely true for x_i' too, since when moving from x_i to x_i' , a 's rank relative to b did not change, whereas it improved relative to c . Thus, x_i is a manipulation of x' .

Thus, in both cases either x_i' is a manipulation of x or x_i is a manipulation of x' , as claimed.

The claim implies that every edge in $\partial_i A$ corresponds to a different pair (x, x') for which the i th coordinate of x' is a profitable manipulation of x . Since each such edge is chosen with probability $\frac{1}{2} \cdot 3^{-n} \cdot \frac{1}{3}$, the total contribution of such pairs to the lower bound on $M_i(F)$ is $\frac{1}{6} \cdot 3^{-n} \cdot \mathbb{E}_x[|\partial_i A(x^{a,b})|]$. A similar contribution comes from the case $(v_{-i}, v_i, v_i') \in \partial_i B$. \square

Summing the two sides of Equation (2) over i , we get:

$$\sum_{i=1}^n M_i(F) \geq \frac{3^{-n}}{6} \cdot \mathbb{E}_{x \in (L_3)^n} \left[\left(|\partial A(x^{a,b})| + |\partial B(x^{a,b})| \right) \right]. \quad (3)$$

Now we are ready to establish the relation between $\sum_i M_i(F)$ and $M^{a,b}(F)$. Recall that Equation (1) above states:

$$M^{a,b}(f) = \mathbb{E}_{x \in (L_3)^n} \left[\frac{|A(x^{a,b})|}{3^n} \cdot \frac{|B(x^{a,b})|}{3^n} \right].$$

By combination of these two equations, the application of the following proposition to the sets $A(x^{a,b})$ and $B(x^{a,b})$ yields the assertion of Lemma 3.2.

Proposition 3.6. For every pair of disjoint sets $A, B \subset \{0, 1, 2\}^n$, we have:

$$|\partial(A)| + |\partial(B)| \geq 3^{-n} |A| |B|.$$

Proof. We start by “shifting” both A and B upward, using a standard monotonization technique (see, e.g., [Fra87]). The shifting is performed by a process of n steps. We denote $A_0 = A$, and for each $i = 1, \dots, n$, at step i we replace A_{i-1} by a set A_i of the same size that is monotone in the i ’th coordinate (which means that if $v \in A_i$ and $v'_i \geq v_i$ then $(v_{-i}, v'_i) \in A_i$). This is done by moving every v with $v_i < 2$ to have $v_i = 2$ if the obtained element is not already in A , and then moving every v that remained with $v_i = 0$ to have $v_i = 1$ if the obtained element is not already in A . Clearly such steps do not change the size of the set, and thus $|A_i| = |A|$ for all i . As usual in such operations, it is not hard to check that the step operation does not increase $\partial_j A$ for any j , and in particular, does not destroy the monotonicity in previous indices (see, e.g., [Fra87] for similar arguments). Hence, the sequence $|\partial_j(A_i)|$ is monotone decreasing in i for all j .

Let A' and B' be the sets we obtain after all n steps. We claim that $A' \setminus A$, the set of “new” elements added in the monotonization process, satisfies

$$|A' \setminus A| \leq |\partial(A)|. \tag{4}$$

Indeed, it is clear that every new element added in the i th step of the monotonization corresponds to either one or two edges in $\partial_i(A_{i-1})$ and these edges are disjoint. Thus, denoting by m_i the number of new elements added in the i th step, we get by the monotonicity of the sequence $|\partial_j(A_i)|$, that:

$$|A' \setminus A| \leq \sum_{i=1}^n m_i \leq \sum_{i=1}^n |\partial_i(A_{i-1})| \leq \sum_{i=1}^n |\partial_i(A)| = |\partial(A)|.$$

Similarly, we have

$$|B' \setminus B| \leq |\partial(B)|. \tag{5}$$

Since both A' and B' are monotone in the partial order of the lattice $\{0, 1, 2\}^n$, they are “positively correlated”, by Harris’ theorem [Har60], or by its better known generalization, the FKG inequality [FKG71]. This means that

$$|A' \cap B'|/3^n \geq |A'|/3^n \cdot |B'|/3^n = |A|/3^n \cdot |B|/3^n.$$

However, by assumption A and B are disjoint and thus $A' \cap B' \subseteq (A' \setminus A) \cup (B' \setminus B)$. Therefore, by Equations (4) and (5), we have:

$$|\partial(A)| + |\partial(B)| \geq |(A' \setminus A) \cup (B' \setminus B)| \geq |A' \cap B'| \geq |A| \cdot |B|/3^n.$$

This completes the proof of the proposition, and thus also of Lemma 3.2. \square

4 Reduction from an SCF with Low Dependence on Irrelevant Alternatives to an Almost Transitive GSWF

In this section we present a reduction which allows to pass from an SCF with low dependence on irrelevant alternatives to a GSWF to which one can apply a quantitative version of Arrow’s impossibility theorem. In order to present the results, we need a few more definitions related to GSWFs and to the quantitative Arrow theorem.

Recall that a GSWF on m alternatives is a function $G : (L_m)^n \rightarrow \{0, 1\}^{\binom{m}{2}}$ which is given the preference orders of the voters, and outputs the preference of the society amongst each pair (a, b) of alternatives. The output preference of G between a and b for a given profile $x \in (L_m)^n$ is denoted by $G^{a,b}(x) \in \{0, 1\}$, where $G^{a,b}(x) = 1$ if a is preferred over b , and $G^{a,b}(x) = 0$ if b is preferred over a . G satisfies the IIA condition if for any pair (a, b) , the function $G^{a,b}(x)$ depends only on the vector $x^{a,b} \in \{0, 1\}^n$ (which represents the preferences of the voters between a and b), and not on other alternatives.

As was shown by Condorcet in 1785, a GSWF based on the majority rule amongst pairs of alternatives can result in a non-transitive outcome, that is, a situation in which there exist alternatives a, b, c , such that a is preferred by the society over b , b is preferred over c , and c is preferred over a . The seminal impossibility theorem of Arrow [Arr50, Arr63] asserts that such non-transitivity occurs in any “non-trivial” GSWF on at least three alternatives satisfying the IIA condition:

Theorem 4.1 (Arrow). *Consider a GSWF G with at least three alternatives. If the following conditions are satisfied:*

- *The IIA condition,*

- *Unanimity* — if all the members of the society prefer some alternative a over another alternative b , then a is preferred over b in the outcome of F ,
- F is not a dictatorship (that is, the preference of the society is not determined by a single member),

then there exists a profile for which the outcome is non-transitive.

Since we would like to use quantitative versions of Arrow's theorem on three alternatives, we use the following notation:

Notation 4.2. For a GSWF G on three alternatives, let

$$NT(G) = \Pr_{x \in (L_3)^n} [G(x) \text{ is non-transitive}].$$

The family of all GSWFs on three alternatives satisfying the IIA condition whose output is always transitive (i.e., those trivial GSWFs for which the conclusion of Arrow's theorem does not apply) was partially characterized by Wilson [Wil72], and fully characterized by Mossel [Mos09]. It consists exactly of all the dictatorships and the anti-dictatorships (i.e., GSWFs whose output is either the preference order of a single voter or its inverse), and the GSWFs which rank a fixed alternative always at the top (or always at the bottom). (See Theorem 6.1 for the exact formulation.) Clearly, all such GSWFs are undesirable from the point of view of Social choice theory, and one may assume that a reasonable GSWF is “far” from being contained in this set. To quantify this notion, we denote

$TR_3 = \{ \text{All GSWFs on 3 alternatives which satisfy IIA and are always transitive} \}$,

and for a GSWF G on three alternatives, denote by

$$Dist(G, TR_3)$$

the minimal fraction of output values that should be changed in order to make G always transitive, while maintaining the IIA condition. The quantitative versions of Arrow's theorem which we use in the next section assert that if $NT(G)$ is small (i.e., G is almost transitive), then $Dist(G, TR_3)$ must be small as well (and thus, G is close to the family of “bad” GSWFs).

Another definition that will be used in the proof is the following:

Definition 4.3. For a GSWF G on m alternatives, and a profile $x \in (L_m)^n$, we say that an alternative a is a Generalized Condorcet Winner (GCW) at profile x if for any alternative $b \neq a$, we have $G^{a,b}(x) = 1$. A Generalized Condorcet Loser (GCL) is defined similarly.

Now we are ready to present our result.

Lemma 4.4. Let $\epsilon_1, \epsilon_2 > 0$, and let F be an SCF on three alternatives, such that:

- $M^{a,b}(F) \leq \epsilon_1$ for all pairs (a, b) .
- F is at least ϵ_2 -far from a dictatorship and from an anti-dictatorship (i.e., an SCF which always outputs the bottom choice of a fixed voter).
- F is at least ϵ_2 -far from breaching non-imposition. That is, for each alternative a , $\Pr_{x \in (L_3)^n} [F(x) = a] \geq \epsilon_2$.

Then one can construct a GSWF G on three alternatives, such that:

1. G satisfies the IIA condition.
2. $\text{Dist}(G, TR_3) \geq \epsilon_2 - 3\sqrt{\epsilon_1}$.
3. $NT(G) \leq 3\sqrt{\epsilon_1}$.

Proof. Given F , we define the GSWF G as follows:

Definition 4.5. For each pair of alternatives a, b , and a profile $x \in (L_3)^n$, we set $G^{a,b}(x) = 1$ if

$$\Pr_{x'} [F(x') = a \mid x'^{a,b} = x^{a,b}] > \Pr_{x'} [F(x') = b \mid x'^{a,b} = x^{a,b}],$$

and $G^{a,b}(x) = 0$ if the reverse inequality holds. In the case of equality we break the tie according to the preference of some fixed voter between a and b .

Intuitively, $G^{a,b}(x)$ considers all profiles x' which agree with x on the preferences of the voters between a and b , and checks whether $F(x') = a$ occurs more often than $F(x') = b$ or the opposite, while ignoring all cases where $F(x')$ equals some other alternative. It is clear from the definition that G satisfies the IIA condition, and that if F is neutral (i.e., invariant under permutation of the alternatives), then G is neutral as well.

In order to analyze G , we introduce an auxiliary definition:

Definition 4.6. A profile $x \in (L_3)^n$ is called a *minority preference on the pair of alternatives (a, b)* if $F(x) = a$ while $G^{a,b}(x) = 0$, or if $F(x) = b$ while $G^{a,b}(x) = 1$. x is called a *minority preference* if it is a minority preference for at least some pair (a, b) . For a fixed pair of alternatives a, b , denote

$$N^{a,b}(F) = \Pr_{x \in (L_3)^n} [x \text{ is a minority preference on } (a, b)].$$

It is easy to relate $N^{a,b}$ to $M^{a,b}$, using the Cauchy-Schwarz inequality:

Proposition 4.7. For every SCF F and every pair of alternatives a, b we have

$$M^{a,b}(F) \geq (N^{a,b}(F))^2.$$

Proof. Given F, a, b , and a vector $x^{a,b} \in \{0, 1\}^n$ representing the preferences of the voters between a and b , define

$$p_a(x^{a,b}) = \Pr_{x'}[F(x') = a \mid x'^{a,b} = x^{a,b}]$$

and

$$p_b(x^{a,b}) = \Pr_{x'}[F(x') = b \mid x'^{a,b} = x^{a,b}].$$

In these terms,

$$M^{a,b}(F) = \mathbb{E}_{x^{a,b} \in \{0,1\}^n} [p_a(x^{a,b}) \cdot p_b(x^{a,b})],$$

while

$$N^{a,b}(F) = \mathbb{E}_{x^{a,b} \in \{0,1\}^n} [\min\{p_a(x^{a,b}), p_b(x^{a,b})\}].$$

Thus, by the Cauchy-Schwarz inequality,

$$M^{a,b}(F) = \mathbb{E}[p_a \cdot p_b] \geq \mathbb{E}[(\min\{p_a, p_b\})^2] \geq (\mathbb{E}[\min\{p_a, p_b\}])^2 = (N^{a,b}(F))^2,$$

as asserted. \square

We are now ready to prove that G satisfies the desired properties.

Consider a profile x that is not a minority preference and denote $a = F(x)$. Note that by the definition of a minority preference, for all b we must have that $G^{a,b}(x) = 1$ and thus, a is a Generalized Condorcet Winner of G at x .

This immediately implies that G satisfies Assertion 3 of the lemma. Indeed,

$$\begin{aligned} NT(G) &= \Pr_x [G \text{ does not have a GCW at } x] \\ &\leq \Pr[x \text{ is a minority preference of } F] \\ &\leq \sum_{a,b} N^{a,b}(F) \leq \sum_{a,b} \sqrt{M^{a,b}(F)} \leq 3\sqrt{\epsilon_1}, \end{aligned}$$

as asserted.

In order to prove Assertion 2, let $Dist(G, TR_3) = \epsilon$, and let $H \in TR_3$ be such that G can be transformed to H by changing only fraction ϵ of the values. We consider four cases:

1. **Case 1: H always ranks alternative a at the top.** In this case, $\Pr[a \text{ is a GCW of } G] \geq 1 - \epsilon$. Note that by the argument above, if x is not a minority preference and a is a GCW of G at x then $F(x) = a$. Hence,

$$\Pr[F(x) = a] \geq (1 - \epsilon) - \Pr[x \text{ is a minority preference of } F] \geq 1 - \epsilon - 3\sqrt{\epsilon_1}.$$

However, by the assumption,

$$\Pr[F(x) = a] \leq 1 - \Pr[F(x) = b] \leq 1 - \epsilon_2,$$

and thus, $\epsilon \geq \epsilon_2 - 3\sqrt{\epsilon_1}$, as asserted.

2. **Case 2: H always ranks alternative a at the bottom.** In this case, $\Pr[a \text{ is a GCL of } G] \geq 1 - \epsilon$. As in the previous case, if x is not a minority preference and a is a GCL of G at x then $F(x) \neq a$. Thus,

$$\Pr[F(x) = a] \leq \epsilon + 3\sqrt{\epsilon_1}.$$

However, by assumption we have $\Pr[F(x) = a] \geq \epsilon_2$, and thus $\epsilon \geq \epsilon_2 - 3\sqrt{\epsilon_1}$, as asserted.

3. **Case 3: H is a dictatorship of voter i .** For a profile x , denote the top alternative in the preference order of voter i by x_i^{top} . We have

$$\Pr[x_i^{top} \text{ is a GCW of } G \text{ at } x] \geq 1 - \epsilon.$$

As in the previous cases, this implies that

$$\Pr[F(x) = x_i^{top}] \geq 1 - \epsilon - 3\sqrt{\epsilon_1}.$$

However, since by assumption, F is at least ϵ_2 -far from a dictatorship of voter i , we have $\epsilon + 3\sqrt{\epsilon_1} \geq \epsilon_2$, and the assertion follows.

4. **Case 4: H is an anti-dictatorship of voter i .** By the same argument as in the previous case, if x_i^{bot} is the bottom alternative in the preference order of voter i , then

$$\Pr[F(x) = x_i^{bot}] \geq 1 - \epsilon - 3\sqrt{\epsilon_1}.$$

However, since F is also at least ϵ_2 -far from anti-dictatorship of voter i , the assertion follows.

This completes the proof of Condition 2 and of Lemma 4.4. \square

Remark 4.8. We note that a certain converse of Lemma 4.4 is true as well. If we have a GSWF G satisfying the IIA condition such that $\Pr[G \text{ has a GCW}] \geq 1 - \epsilon$, then we can define an SCF F to be equal to the GCW of $G(x)$ if such GCW exists, and to the top choice of a fixed voter if the GCW does not exist. Since G satisfies the IIA condition, the event $F(x) = a$ and $F(x') = b$ with $x^{a,b} = x'^{a,b}$ can occur only if either $G(x)$ or $G(x')$ does not have a GCW, and thus, $M^{a,b}(F) \leq 2\epsilon$.

5 Application of a Quantitative Arrow Theorem

The only ingredient required for concluding the proof of Theorem 1.5 is a quantitative version of Arrow’s impossibility theorem. In order to get the optimal result for various assumptions on the SCF F , we use two such versions, due to Kalai [Kal02], and to Keller [Kel10].

Theorem 5.1. *Let G be a GSWF on three alternatives which satisfies the IIA condition. Then:*

1. *If $\text{Dist}(G, TR_3) \geq \epsilon$, then $NT(G) \geq C_1 \cdot \epsilon^3$, where C_1 is a universal constant. [Kel10]*
2. *If, in addition, G is neutral and is at least ϵ -far from a dictatorship and an anti-dictatorship, then $NT(G) \geq C_2 \cdot \epsilon$, where C_2 is a universal constant. [Kal02]*

Now we are ready to present the proof of Theorem 1.5.

Proof. Let F be an SCF on three alternatives, and assume on the contrary that:

- The distance of F from a dictatorship is at least ϵ ,⁶
- For each alternative a , $\Pr[F(x) = a] \geq \epsilon$, but
- $\sum_i M_i(F) < C_0 \cdot \epsilon^6$ (where C_0 is a constant that will be specified below).

By Lemma 3.2, it follows that for each pair of alternatives a, b , we have $M^{a,b}(F) < 6C_0 \cdot \epsilon^6$. By Lemma 4.4, it then follows that there exists a GSWF G on three alternatives which satisfies the IIA condition, and

- $\text{Dist}(G, TR_3) \geq \epsilon - 3\sqrt{6C_0\epsilon^6} \geq \epsilon/2$. (The second inequality holds for C_0 sufficiently small.)
- $NT(G) < 3\sqrt{6C_0\epsilon^6} = 3\sqrt{6C_0} \cdot \epsilon^3$.

However, for C_0 small enough (concretely, $C_0 \leq C_1^2/3456$ where C_1 is the constant in Theorem 5.1), this contradicts the first version of Theorem 5.1 above. This proves the first assertion of Theorem 1.5. The second assertion follows similarly using the second version of Theorem 5.1 instead of the first one (note that by the construction of G , if F is neutral then G is neutral as well and thus Kalai’s version of the quantitative Arrow theorem can be applied). This completes the proof of Theorem 1.5. \square

⁶We note that there is no need to add the condition that F is far from an anti-dictatorship, since an SCF which is close to an anti-dictatorship can be clearly manipulated by the “anti-dictator”.

Remark 5.2. Since the value of the constant C_1 in the first version of Theorem 5.1 is extremely low (i.e., of order $\exp(2^{-10,000,000})$), for certain values of n and ϵ , a better result can be obtained by using another version of the quantitative Arrow theorem. In that version, obtained by Mossel [Mos09], the lower bound $C \cdot \epsilon^3$ is replaced by $(1/36000) \cdot \epsilon^3 n^{-3}$. Applying Mossel’s theorem instead of the version we used above, we get the lower bound

$$\sum_i M_i(F) \geq C' \cdot \epsilon^6 / n^6,$$

where $C' \approx 10^{-8}$. While this bound depends also on n , for certain values of the parameters it is still stronger, due to the bigger value of the constant.

6 SCFs with More than Three Alternatives

In this section we consider SCFs with more than three alternatives. We show that the second step of our proof (reduction from an SCF with low dependence on irrelevant alternatives to an almost transitive GSWF) can be generalized to SCFs on m alternatives, and that the third step (application of a quantitative Arrow theorem) can be generalized under an additional assumption of neutrality. However, we weren’t able to generalize the first step (reduction from low manipulation power to low dependence on irrelevant alternatives), and thus we do not obtain any variant of the main theorem for more than three alternatives.

We would like to mention again two related follow-up works. Xia and Conitzer [XC08] showed that the first step of our proof can be generalized to any constant number of alternatives under several additional assumptions (see Section 2). Furthermore, Isaksson et al. [IKM09] obtained a quantitative Gibbard-Satterthwaite theorem for any number of alternatives under a single additional assumption of neutrality, using a different technique.

Despite these two works, we decided to present the partial generalization of our proof to more than three alternatives, hoping that the technique can be extended to obtain a quantitative Gibbard-Satterthwaite theorem without the neutrality assumption.

6.1 Reduction from an SCF with Low Dependence on Irrelevant Alternatives to a GSWF which Almost Always has a Condorcet Winner

In order to present the results of this section, we have to generalize the notions of TR_3 and $NT(G)$ defined in Section 4 to GSWFs on m alternatives.

The class TR_m of all GSWFs on m alternatives which satisfy the IIA condition and whose output is always transitive, was partially characterized by Wilson [Wil72], and fully characterized by Mossel [Mos09] in the following theorem:

Theorem 6.1 (Mossel). *The class TR_m consists exactly of all GSWFs G on m alternatives satisfying the IIA condition, for which there exists a partition of the set of alternatives into disjoint sets A_1, A_2, \dots, A_r such that:*

- *For any profile, G ranks all the alternatives in A_i above all the alternatives in A_j , for all $i < j$.*
- *For all s such that $|A_s| \geq 3$, the restriction of G to the alternatives in A_s is a dictatorship or an anti-dictatorship.*
- *For all s such that $|A_s| = 2$, the restriction of G to the alternatives in A_s is an arbitrary non-constant function of the individual preferences between the two alternatives in A_s .*

While the notion $NT(G)$ makes sense also for GSWFs on m alternatives, we use here a different notion which coincides with $NT(G)$ in the case of three alternatives:

Notation 6.2. Let G be a GSWF on m alternatives. The probability that G does not have a Generalized Condorcet Winner (GCW) is denoted by

$$NGCW(G) = \Pr_{x \in (L_m)^n} [G \text{ does not have a GCW at } x].$$

Similarly, $GCW(G)$ denotes the probability that G has a GCW.

Under these definitions, Lemma 4.4 generalizes directly to the case of m alternatives. We get:

Lemma 6.3. Let $\epsilon_1, \epsilon_2 > 0$, and let F be an SCF on m alternatives, such that:

- $M^{a,b}(F) \leq \epsilon_1$ for all pairs (a, b) .
- F is at least ϵ_2 -far from a dictatorship and from an anti-dictatorship.
- There exist alternatives a, b, c , such that

$$\min \left(\Pr_{x \in (L_m)^n} [F(x) = a], \Pr_{x \in (L_m)^n} [F(x) = b], \Pr_{x \in (L_m)^n} [F(x) = c] \right) \geq \epsilon_2.$$

Then one can construct a GSWF G on m alternatives, such that:

1. G satisfies the IIA condition.
2. $\text{Dist}(G, TR_m) \geq \epsilon_2 - \binom{m}{2} \cdot \sqrt{\epsilon_1}$.
3. $\text{NGCW}(G) \leq \binom{m}{2} \cdot \sqrt{\epsilon_1}$.

Furthermore, if F is neutral, then G is neutral as well.

Note that the third condition imposed on F , which means that F is ϵ_2 -far from having only two alternatives in its range, is weaker than being ϵ_2 -far from breaching non-imposition (which means that any alternative is elected with probability at least ϵ_2).

The proof of Lemma 6.3 is very similar to the proof of Lemma 4.4, and thus we present only the required modifications.

Sketch of Proof. The definition of G and the proofs of Assertions 1 and 3 are the same as in the proof of Lemma 4.4. In order to prove Assertion 2, let $\text{Dist}(G, TR_m) = \epsilon$, and let $H \in TR_m$ be such that G can be transformed to H by changing only fraction ϵ of the values. By Theorem 6.1 applied to H , the set of alternatives can be partitioned into disjoint sets A_1, A_2, \dots, A_r such that for any profile, H ranks all the alternatives in A_i above all the alternatives in A_j , for all $i < j$. Note that for any alternative $d \notin A_1$, $F(x) = d$ can occur only if either $G(x) \neq H(x)$ or x is a minority preference. Hence,

$$\Pr_x[F(x) = d] \leq \epsilon + \binom{m}{2} \cdot \sqrt{\epsilon_1}.$$

Therefore, either $\epsilon_2 \leq \epsilon + \binom{m}{2} \cdot \sqrt{\epsilon_1}$ (as claimed in Assertion 2 of the lemma), or $a, b, c \in A_1$. In the latter case, $|A_1| \geq 3$ and thus, by Theorem 6.1, the restriction of H to the alternatives in A_1 is a dictatorship or an anti-dictatorship. In both cases, the assertion of the lemma is proved in the same way as cases 3 and 4 in the proof of Lemma 4.4. \square

6.2 Generalization of the Quantitative Arrow Theorem

The quantitative versions of Arrow's theorem presented by Mossel [Mos09] and Keller [Kel10] apply also to GSWFs on m alternatives, and assert that if $\text{Dist}(G, TR_m)$ is not too small, then $\text{NT}(G)$ is also not too small. However, the reduction given by Lemma 6.3 yields a bound on $\text{NGCW}(G)$ rather than on $\text{NT}(G)$, and thus we need a lower bound on $\text{NGCW}(G)$ (the probability of not having a Generalized Condorcet Winner), which may be much lower than the probability of being non-transitive.

In this subsection we prove a generalization of the quantitative Arrow theorem which allows to obtain a lower bound on $NGCW(G)$. However, we require the additional assumption that G is neutral (i.e., invariant under permutation of the alternatives), and our proof relies heavily on this assumption.

Before we present the generalization, we recall a few properties of neutral GSWFs. Let G be a GSWF on n voters and m alternatives denoted by $\{1, 2, \dots, m\}$. If G satisfies the IIA condition, then its output is determined by $\binom{m}{2}$ Boolean functions $G^{i,j} : \{0, 1\}^n \rightarrow \{0, 1\}$, which are given the individual preferences between alternatives i and j and output the preference of the society between them. If, in addition, G is neutral, then all the functions $G^{i,j}$ are equal, and thus we denote them by a single function $g : \{0, 1\}^n \rightarrow \{0, 1\}$, and write $G = g^{\otimes \binom{m}{2}}$. Note that the neutrality assumption implies also that g is an odd function, that is, $g(a_1, \dots, a_n) = 1 - g(1 - a_1, \dots, 1 - a_n)$, and in particular, $\Pr[g = 1] = 1/2$. We denote the distance of g from a dictatorship or an anti-dictatorship by $\text{Dist}(g, \text{DICT}_2)$.

We are now ready to present our result. We start with an equivalent formulation of Kalai's version of the quantitative Arrow theorem [Kal02].

Theorem 6.4 (Kalai). *There exists a constant $C_3 > 0$ such that the following holds. Let $G = g^{\otimes \binom{3}{2}}$ be a neutral GSWF on n voters and 3 alternatives satisfying the IIA condition. Then*

$$NGCW(G) \geq C_3 \cdot \text{Dist}(g, \text{DICT}_2).$$

We prove the following generalization:

Theorem 6.5. *For any $\epsilon > 0$, and for every $m \geq 3$, there exists a constant $\delta_m(\epsilon) > 0$ such that the following holds. Let $G = g^{\otimes \binom{m}{2}}$ be a neutral GSWF on n voters and m alternatives which satisfies the IIA condition. If $\text{Dist}(g, \text{DICT}_2) \geq \epsilon$, then $NGCW(G) \geq \delta_m$.*

Moreover, for $m = 3, 4, 5$, we can take $\delta_m = C \cdot \epsilon$, where C is a universal constant.

Before we present the proof of the theorem, we note that if a neutral GSWF $G = g^{\otimes \binom{m}{2}}$ on m alternatives is at least ϵ -far from a dictatorship and from an anti-dictatorship, then $\text{Dist}(g, \text{Dict}_2) \geq \epsilon / \binom{m}{2}$. Thus, Theorem 6.5 implies immediately the following corollary.

Corollary 6.6. *For any $\epsilon > 0$, and for every $m \geq 3$, there exists a constant $\delta'_m(\epsilon) > 0$ such that the following holds. Let G be a neutral GSWF on m alternatives satisfying the IIA condition. If G is at least ϵ -far from a dictatorship and from an anti-dictatorship, then $NGCW(G) \geq \delta'_m$.*

Moreover, for $m = 3, 4, 5$, we can take $\delta'_m = C \cdot \epsilon$, where C is a universal constant.

Proof of Theorem 6.5. The case $m = 3$ is exactly Kalai's theorem above. We first give a direct proof of the cases $m = 4$ and $m = 5$, and then show a general inductive argument that allows to leverage the result to any $m > 5$.

GSWFs on four alternatives

We begin by considering the case $m = 4$. For $1 \leq i, j \leq 4$, let X_{ij} be the random 0/1 variable that indicates the event $G^{i,j}(x) = 1$, where the profile x is chosen at random. Note that by the neutrality assumption, the probability $GCW(G)$ is precisely four times the probability that alternative 1 is a GCW of G . Hence,

$$GCW(G) = 4 \cdot \mathbb{E}[\prod_{j=2}^4 X_{1j}] = 4 \cdot \mathbb{E}[\prod_{j=2}^4 (1 - X_{j1})]. \quad (6)$$

Before expanding this equation, we make three observations. First, from the neutrality of G it follows that g is balanced (i.e., $\Pr[g = 1] = 1/2$), and thus, for all $j \in \{2, 3, 4\}$ we have

$$\mathbb{E}[X_{j1}] = 1/2.$$

Next, for any pair i, j with $2 \leq i, j \leq 4$, we can apply Kalai's theorem to the GSWF G' which is the restriction of G to the alternatives $\{1, i, j\}$ to get:

$$\begin{aligned} \mathbb{E}[X_{j1} X_{i1}] &= \Pr[1 \text{ is a GCL of } G'] \\ &= \frac{1}{3} \cdot GCW(G') \leq \frac{1}{3}(1 - C_3 \cdot \text{Dist}(g, DICT_2)). \end{aligned}$$

Finally, from neutrality, the probability that alternative 1 is a GCW is precisely equal to the probability that he is a GCL, and thus,

$$\mathbb{E}[\prod_{j=2}^4 X_{1j}] = \mathbb{E}[\prod_{j=2}^4 X_{j1}].$$

Using these observations we expand Equation (6) and get:

$$GCW(G) = 4(1 - 3 \cdot \frac{1}{2} + 3 \cdot \frac{1}{3} \cdot GCW(G') - \frac{1}{4} \cdot GCW(G)),$$

and thus, by Kalai's theorem,

$$GCW(G) = 2GCW(G') - 1 \leq 1 - 2C_3 \cdot \text{Dist}(g, DICT_2), \quad (7)$$

which yields the assertion of the theorem for $m = 4$ with $\delta_4 = 2C_3 \cdot \epsilon$.

GSWFs on five alternatives

Next we consider the case $m = 5$. Unfortunately, the first natural step, generalizing the inclusion-exclusion type formula (6) to get

$$GCW(G) = 5 \cdot \mathbb{E}\left[\prod_{j=2}^5 (1 - X_{j1})\right],$$

does not help, due to an annoying prosaic reason: the two terms $\mathbb{E}[\prod_{j=2}^5 (X_{1j})]$ and $\mathbb{E}[\prod_{j=2}^5 (1 - X_{j1})]$ which appear on the two sides of the equation have the same sign, and cancel out. To remedy this we consider a neutral GSWF G_6 on six alternatives, and denote its restrictions to five, four, and three alternatives by G_5 , G_4 , and G_3 , respectively. We start with the expansion

$$GCW(G_6) = 6 \cdot \mathbb{E}\left[\prod_{j=2}^6 (1 - X_{j1})\right],$$

which gives:

$$\begin{aligned} GCW(G_6) = & 6\left(1 - \frac{5}{2} + \binom{5}{2} \frac{GCW(G_3)}{3} - \right. \\ & \left. - \binom{5}{3} \frac{GCW(G_4)}{4} + \binom{5}{4} \frac{GCW(G_5)}{5} - \frac{GCW(G_6)}{6}\right). \end{aligned}$$

Rearranging, and using Equation (7), we get:

$$\frac{GCW(G_6)}{3} + \frac{5 \cdot GCW(G_3)}{3} - 1 = GCW(G_5).$$

Since $GCW(G_6) \leq 1$ and $GCW(G_3) \leq 1 - C_3 \cdot \text{Dist}(g, DICT_2)$, this yields

$$GCW(G_5) \leq 1 - \frac{5}{3} C_3 \cdot \text{Dist}(g, DICT_2),$$

which is the assertion of the theorem for $m = 5$ with $\delta_5 = \frac{5}{3} C_3 \cdot \epsilon$.

GSWFs on more than five alternatives

The assertion of the theorem for all $m > 5$ follows from the cases $m = 3, 4, 5$ using full induction.

Assume that we already proved the assertion for m_1 and m_2 , and let $G = g^{\otimes \binom{m_1+m_2}{2}}$ be a GSWF on $m_1 + m_2$ alternatives, such that $\text{Dist}(g, Dict_2) = \epsilon$.

Applying the assertion to the restrictions of G to the first m_1 alternatives and to the last m_2 alternatives, we get that with probability at least δ_{m_1} , there is no GCW among the first m_1 alternatives, and with probability at least δ_{m_2} there is no GCW among the last m_2 alternatives. The key point here is that these two events are independent since the voter preferences within two disjoint sets of alternatives are totally independent of each other. Thus, the probability that there is no GCW at all is at least $\delta_{m_1} \cdot \delta_{m_2}$.

Starting with $\delta_m = C \cdot \epsilon$ for $m = 3, 4, 5$, we get $\delta_m \geq (C\epsilon)^{\lfloor m/3 \rfloor}$ for general m . (Luckily, every integer $m > 5$ can be represented as $m = 3a + 4b$, with a and b nonnegative integers.)

This completes the proof of Theorem 6.5. \square

Remark 6.7. We note that the value of δ_m obtained in our proof for general values of m decreases as $\epsilon^{O(m)}$. However, we conjecture that for a fixed $\epsilon > 0$, not only that δ_m need not decrease with m , it actually tends to 1. This conjecture is supported by a recent work of Mossel [Mos10] who calculated the asymptotic value $\lim_m \lim_n [1 - \delta(m)] = \Theta(1/m)$ for the case when $G^{a,b} : \{0, 1\}^n \rightarrow \{0, 1\}$ is a low-influence function (e.g., the majority function) on $x^{a,b}$ for all $a, b \in \{1, 2, \dots, m\}$.

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